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APPLIED FISHERY STATISTICS

Vectors and Matrices

by

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PREFACE

This supplement is the first one in a series of supplements to be issued, enlarging the field of coverage of the manual Applied Fishery Statistics. It gives a brief account of matrices and matrix algebra.

This supplement will be of constant service during the study of subsequent supplements of the manual, covering the application of more advanced statistical methods in the field of applied fishery researches.

CONTENTS

VECTORS AND MATRICES

1. INTRODUCTION

A matrix is any rectangular (or square) array of numbers, and a vector[1] is a special case of a matrix which has only one row or one column. For example, we call vectors the ordered pairs like (x, y), ordered triples like (x, y, z). We call matrices the rectangular arrays like:

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \qquad \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \qquad \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$

or by using another notation, the above matrices can be written:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

2. MATRICES

2.1 Dimension of a Matrix

The numbers of which a matrix is composed are called its elements. According to the above notation the individual a_{ij} denotes an element of the matrix in the i^{th} row (i = 1,2, ... m) and j^{th} column (j = 1,2, ... n). Matrices, usually denoted by the letter A, may be of any dimension (m x n). A matrix of order, m x n is written as follows:

1/ See Section 4

$$A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Example: The following are matrices of various dimensions:

$$A_{2\times 2} = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix} \qquad A_{2\times 3} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad A_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{etc.}$$

2.2 The operations of matrix algebra

Definition - 1:

Two matrices are equal if and only if they have the same dimensions ($m \times n$) and are identical, i.e. each element of A is exactly identical to each element of B.
For example,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

if and only if,

$$a_{11} = b_{11}, \quad a_{12} = b_{12}, \quad a_{21} = b_{21}, \quad a_{22} = b_{22}.$$

Definition - 2:

The sum of two matrices of the same dimensions is a matrix whose elements are the sums of the corresponding elements of the given matrices.
For example,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

Definition - 3:

A matrix (of any pair of dimensions), each of whose elements is zero, is called the null matrix (for that pair of dimensions).
Examples of null matrices are,

$$(0) \qquad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Definition - 4:

The product of a scalar k times a matrix is a matrix whose elements are k times the corresponding elements of the given matrix.
For example,

$$k \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$$

NOTE: Generally matrices are multiplied by real scalar

Definition - 5: Product of matrices

Let A be a $m \times n$-dimensional matrix, and B a $n \times r$-dimensional matrix. Their product $AB = C$ is a $m \times r$-dimensional matrix whose elements are as follows: The element of the i^{th} row and the j^{th} column of C is the inner product[1/] of the i^{th} row-vector of A with the j^{th} column-vector of B.
Examples:

a)

$$A_{3\times 3} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad B_{3\times 1} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$C_{3\times 1} = A_{3\times 3} \times B_{3\times 1} = \begin{pmatrix} a_1x + b_1y + c_1z \\ a_2x + b_2y + c_2z \\ a_3x + b_3y + c_3z \end{pmatrix}$$

b)

$$A_{2\times 3} = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \end{pmatrix} \quad B_{3\times 2} = \begin{pmatrix} 4 & 2 \\ -1 & 3 \\ 2 & 1 \end{pmatrix}$$

$$C_{2\times 2} = A_{2\times 3} \times B_{3\times 2} = \begin{pmatrix} (8 + 1 + 8) & (4 - 3 + 4) \\ (4 - 3 + 10) & (2 + 9 + 5) \end{pmatrix} = \begin{pmatrix} 17 & 5 \\ 11 & 16 \end{pmatrix}$$

1/ See Section 4

c)

$$A_{2\times 2} = \begin{pmatrix} 4 & -1 \\ 2 & 5 \end{pmatrix} \quad B_{2\times 2} = \begin{pmatrix} 4 & 5 \\ -6 & 2 \end{pmatrix}$$

$$C_{2\times 2} = A_{2\times 2} \times B_{2\times 2} = \begin{pmatrix} (16+6) & (20-2) \\ (8-30) & (10+10) \end{pmatrix} = \begin{pmatrix} 22 & 18 \\ -22 & 20 \end{pmatrix}$$

d)

$$D_{1\times 1} = A_{1\times 2} \times B_{2\times 2} \times C_{2\times 1} = (x \quad y) \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

we have,

$$D_{11} = (x + 3y \quad 2x + 4y) \begin{pmatrix} x \\ y \end{pmatrix} = (x^2 + 3xy + 2xy + 4y^2).$$

In view that,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$$

and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y + c_1 \\ a_{21}x + a_{22}y + c_2 \end{pmatrix}$$

we can write the system of simultaneous equations

$$\begin{aligned} a_{11}x + a_{12}y + c_1 &= 0 \\ a_{21}x + a_{22}y + c_2 &= 0 \end{aligned}$$

in the compact form

$$AX + C = 0$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad 0 = \begin{pmatrix} o \\ o \end{pmatrix}$$

As another example, the simultaneous system,

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_2 &= 0 \\ a_3x + b_3y + c_3z + d_3 &= 0 \end{aligned}$$

can be written in the form

$$AX + D = O$$

where,

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \quad O = \begin{pmatrix} o \\ o \\ o \end{pmatrix} \quad \text{etc.}$$

Remarks

1. The product of two matrices AB is defined only when the number of columns in A is equal to the number of rows in B.

2. When A and B are square and are of the same dimension, both AB and BA are defined. In general, however, AB does not equal BA; that is, multiplication of square matrices is not commutative.

   Example:

$$AB = \begin{pmatrix} 3 & 2 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 13 & 5 \\ 21 & -22 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 3 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} -9 & 17 \\ 19 & 5 \end{pmatrix} , \quad AB \neq BA$$

3. The matrix multiplication is associative,

$$A(BC) = (AB)C$$

4. Addition of matrices is commutative, that is,

$$A + B = B + A$$

5. Addition of matrices is associative,

$$A + (B + C) = (A + B) + C$$

Problems

Find the products of the given matrices:

1. $\begin{pmatrix} -2 & 1 & 5 \\ 4 & -1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

2. $\begin{pmatrix} 4 & -2 & 2 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -5 & 2 \\ 1 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 4 & -2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ -2 & 6 \end{pmatrix}$ 4. $\begin{pmatrix} 3 & 6 \\ 4 & 7 \\ 5 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 9 & 10 \end{pmatrix}$

5. $\begin{pmatrix} -10 & 29 & 11 \\ -15 & 45 & 16 \\ 2 & -3 & 5 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & -2 \\ -3 & 2 \end{pmatrix}$

6. Let $A_{2\times 2} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. Compute $A^2 - 5A$

7. Let $B_{3\times 3} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$. Compute $B^2 - 3B$

8. Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & -d \\ a & c \end{pmatrix}$.

Compute $A + B$, AB and BA

9. Write the systems of linear equations in a matrix form,

a) $\begin{aligned} 3x + 5y - 5 &= 0 \\ x - 3y + 7 &= 0 \end{aligned}$ b) $\begin{aligned} x + 5y + 10 &= 0 \\ 2x - 3y + 2 &= 0 \end{aligned}$

c) $\begin{aligned} x + y + 2 &= 0 \\ y - 22 - 1 &= 0 \\ 3x + 2y + 5 &= 0 \end{aligned}$ or, $\begin{aligned} x + y + 2 + 0 &= 0 \\ 0x + y - 22 - 1 &= 0 \\ 3x + 2y - 02 + 5 &= 0 \end{aligned}$

10. Compute the products,

a) $(x \quad y) \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

b) $(x \quad y \quad z) \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

2.3 Rank of a matrix

Rank of a matrix is defined as the number of independent row vectors, so that a non-singular matrix (= a square matrix is called non-singular if its determinant[1] is not equal to zero) is of full rank.

Examples:

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, \quad |A| = 3, \quad \text{and Rank} = 2, \quad (|A| = \text{det.of } A).$$

Since none of the vectors can be expressed as linear combination of others. Hence, A is a non-singular matrix.

Rank of matrix A

The rank of any matrix A is the order of the largest square array in A whose determinant does not varnish. The rank of A is denoted by r(A).

Examples:

(1) $$A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 2 & 0 \end{pmatrix}$$

Since we have the largest square matrix of order 2×2 in it,

$$\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} = -8 \neq 0$$

$$\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} = -4 \neq 0$$

$$\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} = -2 \neq 0$$

the rank of A is $r(A) = 2$.

(2) Let,

$$B = \begin{pmatrix} 2 & 3 & 0 \\ 4 & 6 & 0 \end{pmatrix}$$

Largest square matrix 2×2. Since we have,

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$$

only 1×1 square matrix whose determinant will not be zero, the rank of A is $r(A) = 1$.

1/ See Section 3

(3)
$$A = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 6 & 9 & 12 \end{pmatrix}$$

and

$$|A| = 2\begin{vmatrix} 6 & 9 \\ 9 & 12 \end{vmatrix} - 4\begin{vmatrix} 3 & 9 \\ 6 & 12 \end{vmatrix} + 6\begin{vmatrix} 3 & 6 \\ 6 & 9 \end{vmatrix}$$

$$= 2(72 - 81) - 4(36 - 54) + 6(27 - 36) = 0$$

The matrix A is not a non-singular matrix (it is a singular matrix).

2.4 Inverse of a square matrix

Identity matrix (or unit matrix)

There exists an identity matrix (I) for the multiplication of $n \times n$ square matrices, namely the identity matrix. For example, for 3×3 matrices the identity matrix is given,

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Proof,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

and moreover,

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

For 2×2 matrices, the identity matrix is given by,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ etc.}$$

Definition:

If A is a square matrix, its inverse is a square matrix A^{-1} (read "A inverse") which satisfies the equation

$$AA^{-1} = I$$

NOTE:
In the notation A^{-1} the -1 is not an exponent. Do not write[1/] $A^{-1} = \frac{1}{A}$.
Some square matrices do not have inverse matrices. If,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then, the above matrix A has an inverse matrix A^{-1} ($AA^{-1} = I$) if $ad - bc \neq 0$. If $ad - bc = 0$ the inverse matrix of A does not exist. Specifically, A has an inverse matrix if $|A| \neq 0$, ($|A|$ = determinant A). A 2×2 matrix for which $ad - bc \neq 0$ is called non-singular; if $ad - bc = 0$, it is singular. Hence, a matrix has an inverse if and only if it is non-singular[2/].

Examples:

$$A = \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}$$

If $A^{-1} = \begin{pmatrix} u & x \\ y & z \end{pmatrix}$ is the inverse matrix of A, then

$$AA^{-1} = I$$

or,

$$\begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} u & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus we derive,

$$\begin{array}{lcl} 2u + 5y = 1 & \qquad & 2x + 5z = 0 \\ -u + 4y = 0 & & -x + 4z = 1 \end{array}$$

1/ But $A^{-1} = \frac{I}{A}$, where I is an identity matrix

2/ If two rows (columns) of a matrix are proportional (or equal), its determinant is zero

The above systems of equations are equivalent to:

$$\begin{aligned} 2u + 5y &= 1 \\ -2u + 8y &= 0 \end{aligned} \qquad\qquad \begin{aligned} 2x + 5z &= 0 \\ -2x + 8z &= 2 \end{aligned}$$

From the above first system of equations we get,

$$y = \frac{1}{13} \quad \text{and} \quad u = \frac{4}{13} \,.$$

From the second system of equations we get,

$$z = \frac{2}{13} \quad \text{and} \quad x = -\frac{5}{13} \,.$$

Therefore,

$$A^{-1} = \begin{pmatrix} \frac{4}{13} & -\frac{5}{13} \\ \frac{1}{13} & \frac{2}{13} \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} \frac{4}{13} & -\frac{5}{13} \\ \frac{1}{13} & \frac{2}{13} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Examples</u>:

Find the inverse matrices of,

$$\begin{pmatrix} 5 & 6 \\ -2 & 8 \end{pmatrix}, \quad \begin{pmatrix} 6 & -4 \\ 3 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 22 & -10 \\ 6 & 5 \end{pmatrix}, \quad \begin{pmatrix} 4 & 12 \\ -2 & 1 \end{pmatrix}$$

In the case of 3×3 matrices inverses can be computed by the method just described, but the computations are very tedious. Another approach consists of the following steps1/:

1/ See also Section 3

(a) $$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$ is a given 3×3 matrix

(b) Det. $A = |A| = \Delta$, is the determinant of the matrix

(c) The minor determinants[1/] of the elements of the matrix are

$$\begin{matrix} |A_1| & |B_1| & |C_1| \\ |A_2| & |B_2| & |C_2| \\ |A_3| & |B_3| & |C_3| \end{matrix}$$

(d) The cofactors[1/] of the elements of the matrix are,

$$\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$$

(e) The A^{-1} is the inverse matrix of A and is given by,

$$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = \frac{1}{\Delta} J ,$$

where,
$$J = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

the J is called the adjoint matrix. In general, the adjoint[1/] of an $n \times n$ matrix A is another $n \times n$ matrix $J=(A_{ji})$ in which the elements in the i^{th} row and j^{th} column is the cofactor of the element in the j^{th} row and i^{th} column of A.

1/ See also Section 3

(f)
$$AA^{-1} = A^{-1}A = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Examples</u>:

1. Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ be a 2×2 square matrix

(a) $\text{Det. } A = \Delta = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = 1\times3-2\times0 = 3$

(b) The minor determinants of the elements are,

$$|A_1| = 3, \ |B_1| = 2, \ |A_2| = 0, \ |B_2| = 1$$

(c) The cofactors of the elements are,

$$A_1 = 3, \ B_1 = -2, \ A_2 = 0, \ B_2 = 1$$

(d) The adjoint matrix is,

$$J = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}$$

(e) The inverse matrix is,

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

(f)
$$AA^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 2-\frac{6}{3} & \frac{3}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(g) $$A^{-1}A = I = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -\frac{2}{3}+\frac{2}{3} & \frac{3}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. Let $$A = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 1 & 1 \\ 3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

(a) $$\text{Det. } A = \Delta = \begin{vmatrix} 2 & 1 & -2 \\ 0 & 1 & 1 \\ 3 & 0 & 2 \end{vmatrix} = 13$$

(b) The minor determinants of the elements are,

$$|A_1| = 2, \quad |A_2| = 2, \quad |A_3| = 3$$
$$|B_1| = -3, \quad |B_2| = 10, \quad |B_3| = 2$$
$$|C_1| = -3, \quad |C_2| = -3, \quad |C_3| = 2$$

(c) The cofactors of the elements are,

$$A_1 = 2, \quad A_2 = -2, \quad A_3 = 3$$
$$B_1 = 3, \quad B_2 = 10, \quad B_3 = -2$$
$$C_1 = -3, \quad C_2 = 3, \quad C_3 = 2$$

(d) The adjoint matrix is,

$$J = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 3 \\ 3 & 10 & -2 \\ -3 & 3 & 2 \end{pmatrix}$$

(e) The inverse matrix is,

$$A^{-1} = \frac{1}{\Delta} J = \frac{1}{13}\begin{pmatrix} 2 & -2 & 3 \\ 3 & 10 & -2 \\ -3 & 3 & 2 \end{pmatrix}$$

(f) $$AA^{-1} = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} 2 & 1 & -2 \\ 0 & 1 & 1 \\ 3 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & -2 & 3 \\ 3 & 10 & -2 \\ -3 & 3 & 2 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Exercises</u>:

Calculate the inverse of the matrices given below. Check your inverse in the formulas $AA^{-1}=I$ and $A^{-1}A=I$.

1. $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$

2. $\begin{pmatrix} -4 & 2 \\ 5 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

5. $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$

6. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

7. $\begin{pmatrix} 3 & -2 & 4 \\ 1 & 0 & 2 \\ 5 & 3 & -2 \end{pmatrix}$

8. $\begin{pmatrix} 3 & -5 & 0 \\ 2 & 5 & 2 \\ -3 & 1 & 3 \end{pmatrix}$

9. $\begin{pmatrix} 2 & 2 & -3 \\ 2 & 4 & -1 \\ 1 & 5 & 0 \end{pmatrix}$

10. $\begin{pmatrix} 5 & 3 & 3 \\ -1 & 4 & 2 \\ 3 & 0 & 5 \end{pmatrix}$

<u>Properties of the inverses:</u>

It can be proved that,

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof,

(a) $$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

and, $$A^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

(b) $$B = \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 2 & 1 \end{pmatrix}$$

and, $$B^{-1} = \begin{pmatrix} -\frac{1}{8} & \frac{5}{8} \\ \frac{2}{8} & -\frac{2}{8} \end{pmatrix}$$

(c) $$C = A \times B = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 10 & 13 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ f_1 & f_2 \end{pmatrix}$$

$$\text{Det.}C = \Delta = 2 \times 13 - 10 \times 5 = -24$$

Minor determinants,

$$|E_1| = 13, \quad |E_2| = 10, \quad |F_1| = 5, \quad |F_2| = 2$$

Cofactors,

$$E_1 = 13, \quad E_2 = -10, \quad F_1 = -5, \quad F_2 = 2$$

Adjoint matrix J,

$$J = \begin{pmatrix} 13 & -5 \\ -10 & 2 \end{pmatrix}$$

Inverse matrix C^{-1},

$$C^{-1} = \frac{1}{-24} \begin{pmatrix} 13 & -5 \\ -10 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{13}{24} & \frac{5}{24} \\ \frac{10}{24} & \frac{-2}{24} \end{pmatrix}$$

(d)

$$B^{-1} A^{-1} = \begin{pmatrix} \frac{-1}{8} & \frac{5}{8} \\ \frac{2}{8} & \frac{-2}{8} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-13}{24} & \frac{5}{24} \\ \frac{10}{24} & \frac{-2}{24} \end{pmatrix} = C^{-1}$$

If A is a non-singular matrix, then

$$(A^1)^{-1} = (A^{-1})^1$$

For example,

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \qquad \text{det. } A = \Delta = 1$$

$$A^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

We have,

$$(A^{-1})^1 = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

$$A^1 = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \text{ and } (A^1)^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

Hence,

$$(A^{-1})^1 = (A^1)^{-1}$$

2.5 Applications of matrices to simultaneous equations

The theory of matrices and determinants originates in the necessity of solving simultaneous linear equations and of dealing in a compact notation with linear transformations.

We have seen that the simultaneous system of equations,

$$a_1x + b_1y + d_1 = 0$$
$$a_2x + b_2y + d_2 = 0$$

can be written in the compact form,

$$AX + D = 0$$

where,

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \quad O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Similarly,

$$\begin{aligned} a_1x + b_1y + c_1w + d_1 &= 0 \\ a_2x + b_2y + c_2w + d_2 &= 0 \\ a_3x + b_3y + c_3w + d_3 &= 0 \end{aligned}$$

Can be written in the form,

$$AX + D = 0$$

where,

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ w \end{pmatrix} \quad D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \quad O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This suggests that the problem of solving systems of simultaneous linear equations is really that of solving the matrix equation,

$$AX + D = 0$$

or, $$AX = -D$$

Multiplying both sides on the left by A^{-1}, we have

$$A^{-1}AX = -A^{-1}D$$

or, $$IX = -A^{-1}D, \quad A^{-1}A = I$$

or, $$X = -A^{-1}D$$ is the solution.

A possible method of solution therefore is to compute A^{-1} and then find $-A^{-1}D$. This method is known as Cramer's Rule[1/].

1/ For practical uses the Cramer's Rule is written in a different form; see: G.P. Bazigos (1974) - Applied Fishery Statistics, FAO-FIPS/T.135, p.147-152

<u>Example</u>:

Solve,

$$\begin{aligned} 2x + 5y - 8 &= 0 \\ x - 2y + 6 &= 0 \end{aligned}$$

(a) $$A = \begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}, \quad \text{det. } A = -9$$

(b) $$A^{-1} = -\frac{1}{9}\begin{pmatrix} -2 & -5 \\ -1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$$

(c) $$X = -A^{-1}D = \frac{1}{9}\begin{pmatrix} -2 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -8 \\ 6 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} -14 \\ 20 \end{pmatrix}, \quad \text{and } x = -\frac{14}{9}, \quad y = \frac{20}{9}.$$

<u>Exercises</u>:

Use Cramer's Rule to solve the given systems of equations.

1. $$\begin{aligned} 2x + 4y - 16 &= 0 \\ x + 3y - 12 &= 0 \end{aligned}$$

2. $$\begin{aligned} -2x + 2y - 15 &= 0 \\ 5x + y + 6 &= 0 \end{aligned}$$

3. $$\begin{aligned} 2x - 3y + 10w + 4 &= 0 \\ -2x + 4y - 13w - 5 &= 0 \\ x - 2y + 7w + 2 &= 0 \end{aligned}$$

4. $$\begin{aligned} -2x + 2y - 4w + 10 &= 0 \\ 3x + 2y - 2w - 12 &= 0 \\ -3x + 3y + 2w + 7 &= 0 \end{aligned}$$

5. $$\begin{aligned} 2x + 2y - 8w + 6 &= 0 \\ x + y - 2w &= 0 \\ x + 2y + 3w - 5 &= 0 \end{aligned}$$

2.6 Transposition

If the successive columns of a matrix A are written as successive rows of a new matrix B, then B is called the transpose of A. The i^{th} column of B is the i^{th} row of A, and vice versa. If A is an $m \times n$ matrix, then B is an $n \times m$ matrix.

For example,

$$A = \begin{pmatrix} 3 & 7 & 2 \\ 2 & 1 & 0 \\ 6 & 9 & 6 \\ 1 & 3 & 7 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 12 & 6 & 1 \\ 7 & 1 & 9 & 3 \\ 2 & 0 & 6 & 7 \end{pmatrix}$$

The square matrix is said to be symmetric if it is equal to its transpose. That is, $A_{n \times n}$ is symmetric if and only if $a_{ij} = a_{ji}$, for $i,j = 1,2, \ldots n$.

For example,

$$A_{n \times n} = \begin{pmatrix} 2 & 3 & -4 \\ 3 & 5 & 1 \\ -4 & 1 & 6 \end{pmatrix} \text{ is symmetric.}$$

(principal diagonal).

In a symmetric matrix, pairs of equal elements are situated symmetrically with respect to its principal diagonal.

If $B = -A$, the matrix is said to be skew-symmetric. If so, A must be square and the elements of its principal diagonal must all be zero.

For example,

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix} \text{ is skew-symmetric.}$$

Properties of transposed matrix

If A and B are two matrices of given dimensions, and A' and B' are the transposed matrices, the following equations are valid:

(a) If the sum $C = A + B$ is defined, then $C' = A' + B'$

(b) If the product AB is defined, then $(AB)' = B'A'$

(c) $I' = I$

(d) $(A')' = A$

2.7 Diagonal matrix

A diagonal matrix is a square matrix whose off-diagonal elements, a_{ij} for $i \neq j$, are all equal to zero.

Examples:

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 10 & 0 \\ 0 & 15 \end{pmatrix}.$$

2.8 Idempotend matrix

If A is a square matrix, and if $A = A^2$, then A is known as an idempotend matrix.

For example,

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$A^2 = AA = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \times \frac{1}{2} + \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) & \frac{1}{2} \times \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) \left(\frac{1}{2}\right) \\ -\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \left(-\frac{1}{2}\right) & \left(-\frac{1}{2}\right) \left(-\frac{1}{2}\right) + \frac{1}{2} \times \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} + \frac{1}{4} & -\frac{1}{4} - \frac{1}{4} \\ -\frac{1}{4} - \frac{1}{4} & \frac{1}{4} + \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

2.9 Trace of a matrix

Trace of a matrix is the sum of the diagonal elements of a square matrix. For an example,

$$A = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\text{Trace of } A = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

Note: Rank of a symmetric matrix A = Trace of A

Partitioning matrices

It is often necessary to study some sub-set of elements in a matrix which form a sub-matrix. We are given a matrix A and partion is as follows,

$$A = \left(\begin{array}{cc:c} 1 & 3 & 2 \\ 2 & 5 & 0 \\ \hdashline 4 & 1 & 7 \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where,

$$A_{11} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \quad A_{12} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 4 & 1 \end{pmatrix} \quad A_{22} = \begin{pmatrix} 7 \end{pmatrix}$$

we also have,

$$B = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 4 & 5 \\ 6 & 0 & 1 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

where,

$$B_{11} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \qquad B_{12} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$$

$$B_{21} = \begin{pmatrix} 6 \end{pmatrix} \qquad B_{22} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

We have,

$$C = AB = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 0 \\ 4 & 1 & 7 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 2 & 4 & 5 \\ 6 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 13 & 19 \\ 10 & 22 & 29 \\ 44 & 8 & 20 \end{pmatrix}$$

If we use block multiplication, C should be

$$C = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} (A_{11} \; B_{11} + A_{12} \; B_{21}) & (A_{11} \; B_{12} + A_{12} \; B_{22}) \\ (A_{21} \; B_{11} + A_{22} \; B_{21}) & (A_{21} \; B_{12} + A_{22} \; B_{22}) \end{pmatrix}$$

where,

$$(A_{11}\,B_{11} + A_{12}\,B_{21}) = \begin{pmatrix}1 & 3\\2 & 5\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}2\\0\end{pmatrix}\begin{pmatrix}6\end{pmatrix}$$

$$= \begin{pmatrix}6\\10\end{pmatrix} + \begin{pmatrix}12\\0\end{pmatrix} = \begin{pmatrix}18\\10\end{pmatrix}$$

$$(A_{11}\,B_{12} + A_{12}\,B_{22}) = \begin{pmatrix}1 & 3\\2 & 5\end{pmatrix}\begin{pmatrix}1 & 2\\4 & 5\end{pmatrix} + \begin{pmatrix}2\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}13 & 17\\22 & 29\end{pmatrix} + \begin{pmatrix}0 & 2\\0 & 0\end{pmatrix} = \begin{pmatrix}13 & 19\\22 & 29\end{pmatrix}$$

$$\left(A_{21}\,B_{11} + A_{22}\,B_{21}\right) = \begin{pmatrix}4 & 1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} + \begin{pmatrix}7\end{pmatrix}\begin{pmatrix}6\end{pmatrix} = \begin{pmatrix}2\end{pmatrix} + \begin{pmatrix}42\end{pmatrix} = 44$$

$$\left(A_{21}\,B_{12} + A_{22}\,B_{22}\right) = \begin{pmatrix}4 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\4 & 5\end{pmatrix} + \begin{pmatrix}7\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}8 & 20\end{pmatrix}$$

and,

$$C = \begin{matrix}\begin{pmatrix}18\\10\end{pmatrix} & \begin{pmatrix}13 & 19\\22 & 29\end{pmatrix}\\ \begin{pmatrix}44\end{pmatrix} & \begin{pmatrix}8 & 20\end{pmatrix}\end{matrix}$$

The same result was obtained by direct multiplication.

3. DETERMINANTS

With any square matrix there is associated a number called its determinant. The value of the determinant can be found in the following way:

$$A = \begin{pmatrix}a_1 & b_1\\a_2 & b_2\end{pmatrix}$$ is a square matrix of order 2×2.

Then, the determinant of A is given by,

$$D = |A| = \begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix} = a_1 b_2 - a_2 b_1$$

Similarly, let

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad \text{is a square matrix of order } 3 \times 3$$

The determinant of A is given by,

$$D = |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 D_{11} - b_1 D_{12} + c_2 D_{13}$$

$$= a_1 A_{11} + b_1 A_{12} + c_1 A_{13}$$

where,

D_{ij} is called the minor determinants of the elements a_{ij} and A_{ij} is called the co-factors of the elements a_{ij}. Clearly,

$$A_{ij} = (-1)^{i+j} D_{ij}.$$

The adjoint matrix of a $m \times n$ matrix A is another matrix $J = (A_{ji})$ in which the element in the i^{th} row and j^{th} column is the co-factor of the element in the j^{th} row and i^{th} column of A.

As an example, we have

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 5 \\ 7 & 6 & 4 \end{pmatrix} \quad \text{is a square matrix of order } 3 \times 3$$

The determinant of the matrix is,

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 3 & 0 & 5 \\ 7 & 6 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 5 \\ 6 & 4 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 7 & 4 \end{vmatrix} + 0\begin{vmatrix} 3 & 0 \\ 7 & 6 \end{vmatrix}$$

$$= 2\left(0\times 4 - 6\times 5\right) - 1\left(3\times 4 - 7\times 5\right) + 0\left(3\times 6 - 7\times 0\right)$$

$$= -60 + 23 + 0 = -37$$

Also, we have

$$D_{11} = -30, \quad D_{12} = -23, \quad D_{13} = 18$$

$$|A| = 2\times(-30) - 1\times(-23) + 0\times(18) = -37$$

Further, we have

$$A_{11} = (-1)^{1+1}D_{11} = -30, \quad A_{12} = (-1)^{1+2}D_{12} = 23$$

$$A_{13} = (-1)^{1+3}D_{11} = 18$$

$$|A| = 2\times(-30) + 1\times 23 + 0\times 18 = -37$$

Similarly,

$$D_{21} = \begin{vmatrix} 1 & 0 \\ 6 & 4 \end{vmatrix} = 4, \; D_{22} = \begin{vmatrix} 2 & 0 \\ 7 & 4 \end{vmatrix} = 8, \; D_{23} = \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix} = 5$$

$$D_{31} = \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix} = 5, \; D_{32} = \begin{vmatrix} 2 & 0 \\ 3 & 5 \end{vmatrix} = 10, \; D_{33} = \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

Hence,

$$A_{21} = (-1)^{2+1}D_{21} = -4, \; A_{22} = (-1)^{2+2}D_{22} = 8, \; A_{23} = (-1)^{2+3}D_{23} = -5$$

$$A_{31} = (-1)^{3+1}D_{31} = 5, \; A_{32} = (-1)^{3+2}D_{32} = -10, \; A_{33} = (-1)^{3+3}D_{33} = -3$$

The adjoint of the matrix A is given by,

$$J = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} -30 & -4 & 5 \\ 23 & 8 & -10 \\ 18 & -5 & -3 \end{pmatrix}$$

Properties of the determinants

1. If every element of a column or of a row of a determinant is zero, then the value of the determinant is zero,

$$|A| = \begin{vmatrix} 0 & 7 & 10 \\ 0 & 8 & 12 \\ 0 & 9 & 14 \end{vmatrix} = 0, \qquad |A| = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 6 & 8 & 10 \end{vmatrix} = 0$$

2. The value of the determinant is not changed if corresponding columns and rows are interchanged,

$$|A| = \begin{vmatrix} 3 & 4 & 5 \\ 4 & 6 & 7 \\ 1 & 2 & 3 \end{vmatrix} = 3(18-14) - 4(12-7) + 5(8-6) = 2$$

$$|A'| = \begin{vmatrix} 3 & 4 & 1 \\ 4 & 6 & 2 \\ 5 & 7 & 3 \end{vmatrix} = 3(18-14) = 4(12-10) + 1(28-30) = 2$$

3. If $|B|$ is the determinant formed by interchanging two columns or rows in $|A|$, then $|B| = -|A|$,

$$|A| = \begin{vmatrix} 3 & 4 & 5 \\ 4 & 6 & 7 \\ 1 & 2 & 3 \end{vmatrix}$$ interchanging 1^{st} and 2^{nd} rows, we have

$$|B| = \begin{vmatrix} 4 & 6 & 7 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{vmatrix}$$

The calculated values of $|A|$ and $|B|$ are,

$$|A| = 3(18-14) - 4(12-7) + 5(8-6) = 2$$

$$|B| = 4(12-10) - 6(9-5) + 7(6-4) = -2$$

Hence,

$$|B| = -|A|.$$

4. If two rows or columns of a determinant are identical, the value of the determinant is zero,

(a) $$|A| = \begin{vmatrix} 4 & 6 & 7 \\ 4 & 6 & 7 \\ 1 & 2 & 3 \end{vmatrix} = (18-14) - 6(12-7) + 7(8-6)$$
$$= 16-30-14 = 0$$

(b) $$|A| = \begin{vmatrix} 3 & 3 & 5 \\ 5 & 5 & 1 \\ 7 & 7 & 3 \end{vmatrix} = 3(15-7) - 3(15-7) + 5(35-35)$$
$$= 24-24+0 = 0$$

5. If every element of a column or of a row of a determinant is multiplied by a fixed number k, then the value of the determinant is multiplied by k,

$$|A| = \begin{vmatrix} 3 & 4 & 5 \\ 4 & 6 & 7 \\ 1 & 2 & 3 \end{vmatrix} = 2$$

Let first column be multiplied by 4

$$\begin{vmatrix} 3\times 4 & 4 & 5 \\ 4\times 4 & 6 & 7 \\ 1\times 4 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 12 & 4 & 5 \\ 16 & 6 & 7 \\ 4 & 2 & 2 \end{vmatrix}$$

$$= 12(18-14) - 4(48-28) + 5(32-24)$$

$$= 48-80+40 = 88-80 = 8 = (2\times 4)$$

6. The value of a determinant is not changed if, to every element of a column or row, we add k times the corresponding element of another column or row,

$$|A| = \begin{vmatrix} 3 & 4 & 5 \\ 4 & 6 & 7 \\ 1 & 2 & 3 \end{vmatrix} = 2$$

Adding 4 times of the element of the 2^{nd} column to the i^{th} column,

$$\begin{vmatrix} 3+4\times4 & 4 & 5 \\ 4+4\times6 & 6 & 7 \\ 1+4\times2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 19 & 4 & 5 \\ 28 & 6 & 7 \\ 9 & 2 & 2 \end{vmatrix}$$

$$= 19(18-14) - 4(84-63) + 5(56-54)$$

$$= 76 - 84 + 10 = 2$$

7. If A,B are matrices of the order $n\times n$, and if $C = AB$, then

$$|C| = |A|\cdot|B|$$

or,

$$|AB| = |A|\cdot|B|$$

Example:

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 6 \\ 3 & 2 \end{pmatrix}$$

$$C = AB = \begin{pmatrix} 11 & 18 \\ 13 & 14 \end{pmatrix},$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5, \qquad |B| = \begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} = -16$$

$$|C| = \begin{vmatrix} 11 & 18 \\ 13 & 14 \end{vmatrix} = -80,$$

and

$$|A|\cdot|B| = 5 \times (-16) = -80$$

4. VECTORS

To simplify our discussion, we shall suppose that all our vectors have the origin 0 as their initial point, so that the coordinates of their end points are equal to the components of the vectors. In a plane, any vector is determined by the ordered pair of numbers (a,b). In the same way, vectors in space have three components and are determined by a triple (a,b,c)

Using our notation, we shall write our vectors as "row vectors": (a,b) or (a,b,c), or as "column vectors":

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Definition-1:

To add two vectors of the same dimension, we add their corresponding components, i.e.

$$(a,b) + (c,d) = (a+c,\ b+d)$$

or,

$$(a,b,c) + (d,e,f) = (a+d,\ b+e,\ c+f)$$

The figure below gives a geometrical interpretation of the addition of vectors P(a,b) and Q(c,d):

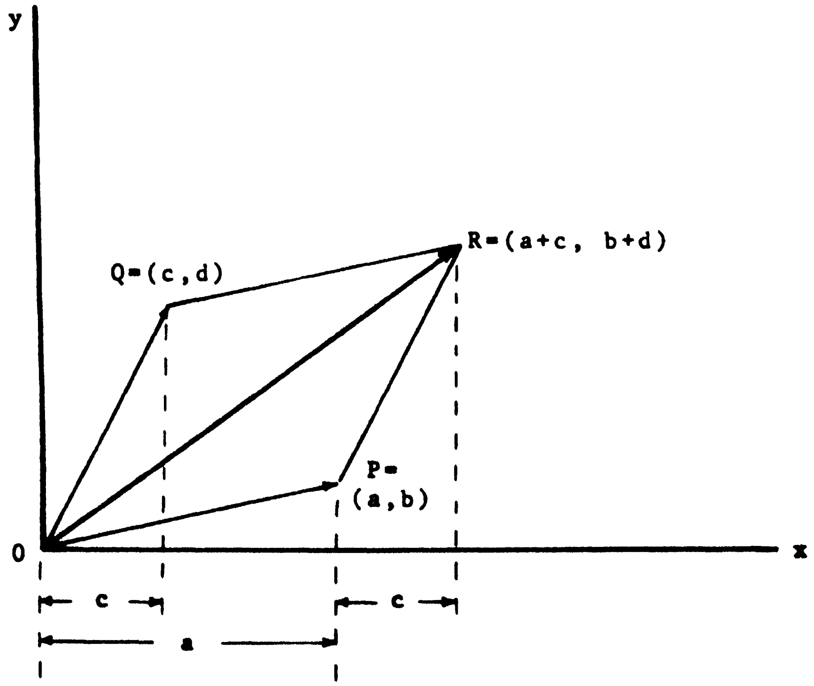

Figure 4.1 The sum of two vectors is the vector obtained by completing the parallelogram

Examples:

1. $(1, -3, 2) + (3, 4, -2) = (4, 1, 0)$

2. $(5, 3, -1) + (-5, -3, +1) = (0, 0, 0)$

3. $(3, 5) - (4, 5) = (-1, 0)$

4. $(-4, 8, 3) - (2, -2, 4) = (-6, 10, -1)$

5. $(6, 3, 4) - (6, 3, 4) = (0, 0, 0)$

4.1 Products of vectors

Definition-1:

If (a,b,c) is a vector and k is a scalar, the product is defined to be the vector, (ka,kb,kc).

Examples:

1. $3(1, 3, -4) = (3, 9, -12)$

2. $-2(1, 3, 2) = (-2, -6, -4)$

3. $0(a, b, c) = (0, 0, 0)$

It is often useful to define three base vectors, i,j,k, as follows. These are vectors of length 1 drawn along the positive directions of the three coordinate axes,

$$i=(1,0,0), \quad j=(0,1,0), \quad k=(0,0,1)$$

In terms of these, any vector (a,b,c) can be written

$$(a,b,c) = (ai + bj + ck)$$

Definition-2:

The inner product of two vectors (a_1,b_1,c_1) and (a_2,b_2,c_2) is defined to be the scalar $a_1a_2 + b_1b_2 + c_1c_2$. This product is denoted by a dot, so that

$$(a_1,b_1,c_1)\ (a_2,b_2,c_2) = a_1a_2 + b_1b_2 + c_1c_2$$

Examples:

1. $(4, 2, -2)\cdot(1, 2, 3) = (4)(1)+(2)(2)+(-2)(3) = 4+4-6 = 2$

2. $(6, 7, 8)\cdot(1, 1, 1) = 6+7+8 = 21$

3. $(-5, 2, 7)\cdot(0, 0, 0) = 0+0+0 = 0$

4. $i\cdot j = (1, 0, 0)(0, 1, 0) = 0+0+0 = 0$

Definition-3:

The length of a vector (a,b,c) is the positive square root of the inner product $(a,b,c)\cdot(a,b,c)$. That is,

$$\text{Length of } (a,b,c) = \sqrt{(a,b,c)\cdot(a,b,c)} = \sqrt{a^2+b^2+c^2}$$

Examples:

Find the length of the vectors given below:

1. (1, 2, 2)

2. (2, 1, 0)

3. (-3, 2, 1)

Statements:

1. Addition of vectors is commutative

2. Addition of vectors is associative

3. The zero vector (0,0,0) is the additive identity for vectors

4. The vector (-a,-b,-c) is the additive inverse of (a,b,c)

5. The inner product is commutative

4.2 Linear combination

A vector $\bar{x}$ in E^n is a linear combination of the vectors $\bar{x}_1$, $\bar{x}_2$, .. $\bar{x}_p$ in E^n, if

$$\bar{x} = a_1\bar{x}_1 + a_2\bar{x}_2 + \ldots + a_p\bar{x}_p$$

for some sets of scalars a_i, $i = 1,2, \ldots, p$.

4.3 Unit vectors

A vector whose i^{th} component is one and all of whose other components are zero is called the unit vector $\bar{e}_i$. If we deal with n component vectors, then there are n such unit vectors,

$$\bar{e}_1 = (1,0,0, \ldots, 0,0)$$

$$\bar{e}_2 = (0,1,0, \ldots, 0,0)$$

$$\bar{e}_n = (0,0,0, \ldots, 0,1)$$

It can be that the unit vectors are each a single column of an identity matrix.

4.4 Null vectors

A vector, all of whose components are zero, is called the null vector. We designated the null vectors by $\bar{o}$, therefore

$$\bar{o} = (0,0,0, \ldots, 0)$$

4.5 Linear independence and orthogonality 1/

A set of vectors $\bar{x}_1,\bar{x}_2, \ldots, \bar{x}_p$ are said to be linearly independent if none of them can be expressed as a linear combination of the rest. For instance, the vectors

$$\bar{x}_1 = (1,0,1,-1,2)$$

$$\bar{x}_2 = (3,2,-1,1,2)$$

$$\bar{x}_3 = (9,4,1,-1,10)$$

1/ I would like to acknowledge the assistance of Mr. M.A. Rahim in the preparation of Sections 4.5 and 4.6

are not independent, since the last vector is the sum of 3 times of the first and 2 times of the second. In general, a set of vectors $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p$ in E^n is said to be linearly dependent if there exists a set of scalars a_i, $i = 1,2, \ldots, p$ - not all of which zero - such that,

$$a_1\bar{X}_1 + a_2\bar{X}_2 + \ldots, + a_p\bar{X}_p = 0$$

In our example,

$$3\bar{x}_1 + 2\bar{x}_2 - 1\bar{x}_3 = 0$$

Hence the set of the above vectors $(\bar{X}_1, \bar{X}_2, \bar{X}_3)$ is dependent. Then, the set of vectors is said to be independent if and only if all $a_i = 0$. For instance,

$$\bar{X}_1 = (1,0,0,0)$$

$$\bar{X}_3 = (0,1,0,0)$$

$$\bar{x}_4 = (0,0,1,0)$$

$$\bar{x}_5 = (0,0,0,1)$$

Orthogonal vectors:

Two non-null vectors are said to be orthogonal if their product is zero. A set of orthogonal vectors with real elements are necessarily independent.

Vector space:

The totality of all vectors obtained by linear combination of a set of vectors is called a vector space.

Such a totality can be generated by a set of independent vectors called a basis of the vector space.

4.6 Rank of a set of vectors

A convenient way of finding the rank of a set of vectors space is by a method known as "sweep-out" consisting of the following procedure:

1. Any vector having a non-zero value for the first element is taken, and its elements are divided by the first so that the resulting vector is of the form

$$(1,\ c_2,\ \ldots,\ c_n)$$

If the elements of the first column are all zero, then it is to be omitted to start with.

From every other vector is subtracted a vector obtained by multiplying $(1, c_2, \ldots, c_n)$ by the first element of the former vector, so that the resulting vectors except the one chosen in (1) above have zero as their first element. The first column is said to be sweeped out by the vector called Pivotal row chosen above.

3. Omission of the Pivotal row and the first column results in a reduced matrix on which operation (1) and (2) repeated until a single non-zero row or all null rows are left over. A single non-zero row left over may be regarded as the last Pivotal row, in which case the rank of the matrix or the rank of the vector space is equal to the number of Pivotal rows.

<u>Example</u>:

(1)	0	1	2	3	
(2)	2	-1	5	4	
(3)	4	0	6	1	
(4)	0	-2	4	7	
(5)	1	$-\frac{1}{2}$	$\frac{5}{2}$	2	, (2) ÷ 2, 1st Pivotal row
(6)		1	2	3	, (1) - (5) × 0
(7)		2	-4	-7	, (3) - (5) × 4
(8)		-2	4	7	, (4) - (5) × 0
(9)		1	2	3	, 2nd Pivotal row
(10)			-8	-13	, (7) - (9) × 2
(11)			8	13	, (8) - (9) × (-2)
(12)			1	$\frac{13}{8}$	, (10) ÷ -8, 3rd Pivotal row

The rank of the vector space is therefore 3, the independent vectors

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$$\frac{3H + 1H}{58}$$